

# Factorizing numbers with classical interference: several implementations in optics

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(Dated: November 13, 2008)

Truncated Fourier, Gauss, Kummer and exponential sums can be used to factorize numbers: for a factor these sums equal unity in absolute value, whereas they nearly vanish for any other number. We show how this factorization algorithm can emerge from superpositions of classical light waves and we present a number of simple implementations in optics.

PACS numbers:

## I. INTRODUCTION

Factorization of numbers into their prime factors is a hard non-polynomial problem for classical computers. It was Shor [1] who proposed a quantum algorithm which can solve the problem of factorization of numbers on a quantum computer with a tremendous speedup as compared to a classical computer. A practical demonstration of Shor's algorithm has been carried out by factorizing the integer 15 [2], using nuclear magnetic resonance. However, quantum computers capable of implementing Shor's algorithm for larger numbers have not been developed yet.

Several approaches to factorize numbers based on interference of multiple quantum paths have been proposed [3, 4, 5, 6, 7, 8, 9, 10, 11]. Those schemes do not use quantum entanglement and do not capitalize on quantum parallelism. As a consequence, these schemes scale exponentially with the number of digits of the factorized number. This is in contrast to Shor's algorithm which requires only a polynomial number of operations. Nevertheless, if the interference is implemented in a suitable way in a system, which does the factorization, then one can benefit because nature plays the role of a computer.

As was pointed out by Jones [12] the proposed techniques for factorization based on Gauss sums [6, 7, 8] unfortunately do not provide useful methods to factorize numbers, because a precalculation of the factors is needed for the experiment. In spite of that, Gauss sums would be useful if it is possible to avoid explicit precalculation stages of the algorithms.

Physical systems that can implement the Gauss sums must be described by complex numbers. In the present paper we investigate how truncated Fourier sum and its generalizations, like truncated Gauss, Kummer and exponential sums, could emerge from superposition of several oscillations. Those sums can be used successfully to factorize numbers. Due to the wide use of interferences and beats in optics, we shall keep our consideration close to optics and our examples are in wave optics too.

However, the proposed implementation can be extended to virtually any physical system where superposition among several different oscillations appear, from the mechanical pendulum with several degrees of freedom, through the atomic and solid state systems and their analogs in quantum mechanics.

## II. TRUNCATED FOURIER, GAUSS, KUMMER AND EXPONENTIAL SUMS

In order to find the factors of a given number  $N$  we use the following truncated sum:

$$\mathcal{A}_N^{(M)}(l) = \frac{1}{M} \sum_{m=1}^M \exp\left(-2\pi i m^k \frac{N}{l}\right), \quad (1)$$

where  $k$  is an integer and  $M$  is the number of terms in the sum. The argument  $l$  scans through all integers between 1 and  $\sqrt{N}$  for possible factors. The capability of the sum of Eq. (1) to factor numbers originate from the fact that for an integer factor  $q$  of  $N$  with  $N = ql$ , all phases in  $\mathcal{A}_N^{(M)}(l)$  are integer multiples of  $2\pi$ . Consequently, the terms add up constructively and yield  $\mathcal{A}_N^{(M)}(l) = 1$ . When  $l$  is not a factor, the phases oscillate rapidly with  $m$ , and  $\mathcal{A}_N^{(M)}(l)$  takes on small values. In this interference pattern, larger truncation parameter  $M$  leads to better convergency. In principle, already the first several terms of the sum are sufficient to discriminate factors from non-factors. Depending

on the coefficient  $k$  in Eq.(1) we distinguish several important cases:

$$\text{Fourier sum for } k = 1 [10], \quad (2a)$$

$$\text{Gauss sum for } k = 2 \quad [5, 6, 7, 8, 9, 10, 13, 14], \quad (2b)$$

$$\text{Kummer sum for } k = 3 [10], \quad (2c)$$

$$\text{exponential sum for } k = m [10]. \quad (2d)$$

The use of quadratic phases to factor numbers (Gauss sum) has the advantage of fewer terms needed in the sum to distinguish factors from non-factors compared to the linear phase (Fourier sum), which is because of high quasi-randomness for the quadratic phase [10]. In the very same way the Kummer sum, and sums with nonlinear phases of higher order, has an advantage compared to the Gauss sum [10].

Now we consider a system with  $M$  different oscillation modes, with frequencies  $\omega_m$ , phases  $\varphi_m$  and amplitudes  $E_{0m}$

$$E_m(t) = E_{0m} \exp(i\omega_m t + i\varphi_m). \quad (3)$$

Using the superposition principle we can write the resulting oscillation as the sum of all oscillations:

$$E(t) = \sum_{m=1}^M E_m(t) = \sum_{m=1}^M E_{0m} \exp(i\omega_m t + i\varphi_m). \quad (4)$$

In the sum of Eq. (4) we can vary the parameters  $E_{0m}$ ,  $\omega_m$ ,  $\varphi_m$  and the time  $t$ . In the next several sections we will show how truncated Fourier, Gauss, Kummer and exponential sums could emerge when we fix three of the parameters for all oscillations, while changing the fourth parameter.

### III. FACTORIZATION USING DIFFERENCES IN TIME DELAY (INTERFEROMETRY)

#### A. Mach-Zehnder interferometer

First we consider the case when the parameters  $E_{0m}$ ,  $\omega_m$ ,  $\varphi_m$  are equal for all oscillations in Eq. (3)

$$E_{0m} = E_0, \quad (5)$$

$$\omega_m = \omega, \quad (6)$$

$$\varphi_m = 0, \quad (7)$$

thus the only parameter that is left not fixed in Eq. (3) is the time  $t$ . This can be easily realized in optics by interferometry, where the individual oscillations describe the electric field for the different arms of the interferometer as shown in Fig.1.

From Fig. 1 and Eq. (4) we see that we have the following sums of electric fields in the detector

$$E = E_0 \sum_{m=1}^M \exp(i\phi_m), \quad (8)$$

where  $\phi_m$  is the phase accumulated in the  $m$  arm of the interferometer due to the difference in travel time through each arm.

Suppose that each arm of the interferometer is with length  $L$ , the wave length of the light that we use in vacuum is  $\lambda$  and the corresponding frequency is  $\omega$ , let the index of refraction in each arm of the interferometer is different and is denoted as  $n_m$ . Then the phase  $\phi_m$  for the beam that travels through the  $m$ -th arm of the interferometer is given as

$$\phi_m = t_m \omega = \frac{L}{c_m} \omega, \quad (9)$$

here  $t_m$  is the time that light travel in the  $m$ -th arm of the interferometer to pass length  $L$  and  $c_m$  is the speed of light in that arm. The refraction index in arm  $m$  of the interferometer is

$$n_m = \frac{c}{c_m}, \quad (10)$$

thus

$$\phi_m = \omega \frac{L}{c} n_m = \frac{2\pi}{\lambda} n_m L, \quad (11)$$

then the electric field in the detector is

$$E = E_0 \sum_{m=1}^M \exp \left( 2\pi i \frac{L}{\lambda} n_m \right). \quad (12)$$

Now if the index of reflection in the  $m$  arm of the interferometer is

$$n_m = a + bm^k, \quad (13)$$

then

$$E = E_0 \exp \left( 2\pi i \frac{aL}{\lambda} \right) \sum_{m=1}^M \exp \left( 2\pi i m^k \frac{bL}{\lambda} \right). \quad (14)$$

The detector registers the intensity,

$$I \sim |E|^2 = \left| \sum_{m=1}^M \exp \left( 2\pi i m^k \frac{bL}{\lambda} \right) \right|^2. \quad (15)$$

Various  $k$  gives us a different type of truncated sum (see Eq. (2)). The number that we want to factorize is  $bL$ , the trial factors are  $\lambda$ . Each time when the trial factor  $\lambda$  is a factor of  $bL$  we will observe a maximum signal in the detector. The number of the terms in the sum can be controlled by doubling the elements in the interferometer Fig. 1. The numbers that could be factorized in this way are of order  $L/\lambda \sim \frac{1m}{1000nm} = 10^6$ .

## B. Pulse train

Now we consider a train of pulses, where the delay of the  $m$  pulse compared to first pulse is given as

$$t_m = m^k \tau, \quad (16)$$

here  $m$  takes the values  $m = 1, 2, 3 \dots M$ , while  $\tau$  can be set as a unit of time.

We consider the case when all pulses have equal amplitudes  $E_{0m} = E_0$  and equal frequencies  $\omega_m = \omega$ . Then the electric field for the  $m$  pulse is given by Eq. (3) and reads

$$E_m = E_0 \exp(i\omega t_m + i\varphi_m) = E_0 \exp(i\omega m^k \tau + i\varphi_m) \quad (17)$$

Let us make a different path way for every pulse in such a way that all pulses hit the same detector at the same time, this is equivalent to make  $\varphi_m = 0$  at the place where all pulses collides. Then the intensity that the detector registers is a result from the superposition among all electric fields, e.g the sum from Eq. (4):

$$I \sim |E|^2 = \left| \sum_{m=1}^M E_m \right|^2 = \left| E_0 \sum_{m=1}^M \exp(2i\pi m^k \nu \tau) \right|^2, \quad (18)$$

where  $\nu = \omega/(2\pi)$ . If one chooses the frequency  $\nu$  as the number that we want to factorize ( $N$ ) and  $1/\tau$  as a trial factor ( $l$ ), then Eq.(18) reduces to the sum from Eq. (1).

## IV. FACTORIZATION USING DIFFERENCES IN FREQUENCIES (BEATS)

If we now consider a system that exhibits several oscillations with the same amplitude  $E_0$  and the same initial phases ( $\varphi_m = 0$ ), but with different frequencies  $\omega_m$ , then the individual oscillations (3) are described by

$$E_m(t) = E_0 \exp(i\omega_m t), \quad (19)$$

$$\omega_m = m^k \omega_0, \quad (20)$$

the resulting oscillation (4) is

$$E(t) = \sum_{m=1}^N E_0 \exp(-2\pi i m^k \nu_0 t), \quad (21)$$

where  $\nu_0 = \omega_0 / (2\pi)$ . We will observe beats when  $t\nu_0$  is a integer which could be used to find the factors of the number  $\nu_0$ . One physical realization of the above idea could be a light with several high-harmonic generated frequencies [15, 16], chosen in the way that they present for example the odd terms in the Fourier sum:

$$\omega_0, 3\omega_0, 5\omega_0, 7\omega_0, \dots \quad (22)$$

then in the detector the time of the detection play the role of the test factors and whenever there is a beat we observe a maximum of the signal, thus this time is a real factor.

## V. FACTORIZATION USING FARADAY EFFECT

The last parameter that we can vary in Eq. (3) is the amplitude of the individual oscillation. For example if we work with laser light we can use the different polarization orientations of the electric field. The electric field is a vector in the polarization plane, which can be described by complex electrical field.

Let us consider the case when we have a linearly polarized light pulse, which is split in several parts and each part passes different pathways through Faraday cells as shown in Fig. 2.

Applying different Faraday rotation angles  $\varphi_m$  on each pathway and collecting all of the light at the same place (at the detector) the resulting electric field is the superposition:

$$E = \frac{E_0}{M} \sum_{m=1}^M \exp(i\varphi_m), \quad (23)$$

where  $E_0$  is the electrical field amplitude of the initial beam. The relation between the angle of polarization rotation due to the Faraday effect  $\varphi_m$  and the magnetic field  $B_m$  in a diamagnetic material [17] is

$$\varphi_m = 2\pi b L B_m, \quad (24)$$

where  $L$  is the length of each pathway and  $2\pi b$  is the Verdet constant for the material [17]. For the amplitude of the resulted electric field we have:

$$E = \frac{E_0}{M} \sum_{m=1}^M \exp(2\pi i b L B_m). \quad (25)$$

If we now have a magnetic field  $B_m$  for the  $m$ -th Faraday cell, which is given as:

$$B_m = B_0 m^k, \quad (26)$$

then the intensity in the detector is

$$I \sim |E|^2 = \left| \frac{E_0}{M} \sum_{m=1}^M \exp(2\pi i m^k b L B_0) \right|^2. \quad (27)$$

Here the number that we want to factorize is  $bL$ , the trial factors are  $1/B_0$ .

## VI. CONCLUSIONS

We have shown how the factorization algorithm based on truncated Fourier, Gauss, Kummer or exponential sums emerges naturally from superpositions of classical light waves. We have proposed a number of simple implementations in optics. These implementations can be extended to virtually any physical system where superpositions of several different oscillations appear.

The factorization algorithms discussed in this paper are classical algorithms and thus their complexity scales exponentially with the number of digits. If an extension of this algorithm exists in entangled quantum systems, then a quantum computing parallelism would be involved with an exponential speedup of factorization. The present solutions therefore could be the first step to an alternative quantum factorization algorithm to the famous Shor algorithm.

### Acknowledgments

This work has been supported by the EU ToK project CAMEL, the EU RTN project EMALI, the EU ITN project FASTQUAST, and the Bulgarian National Science Fund Grants No. WU-2501/06 and No. WU-2517/07. The author is grateful to N. Vitanov for stimulating discussions and critical reading of the manuscript. During the preparation of this paper, the author became aware of a related work by Tamma et al. [18].

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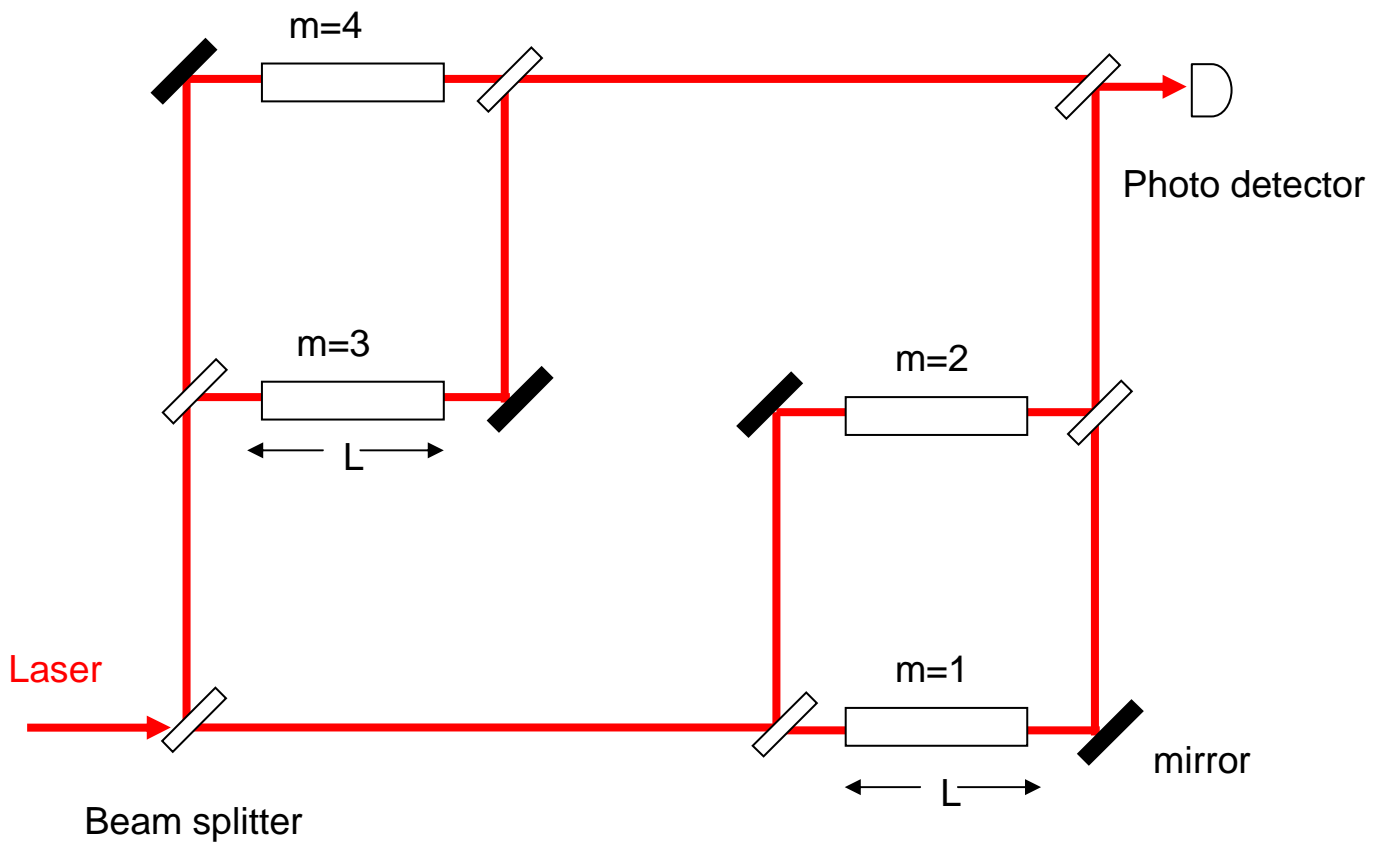


FIG. 1: Four arms Mach-Zehnder interferometer that can be used to factorize numbers by using Fourier, Gauss, Kummer or exponential sums. The four arms correspond to four terms in the sum. Repeating the procedure for doubling the arm in principle one can increase the terms in the sum as much as needed to distinguish factors from nonfactors.

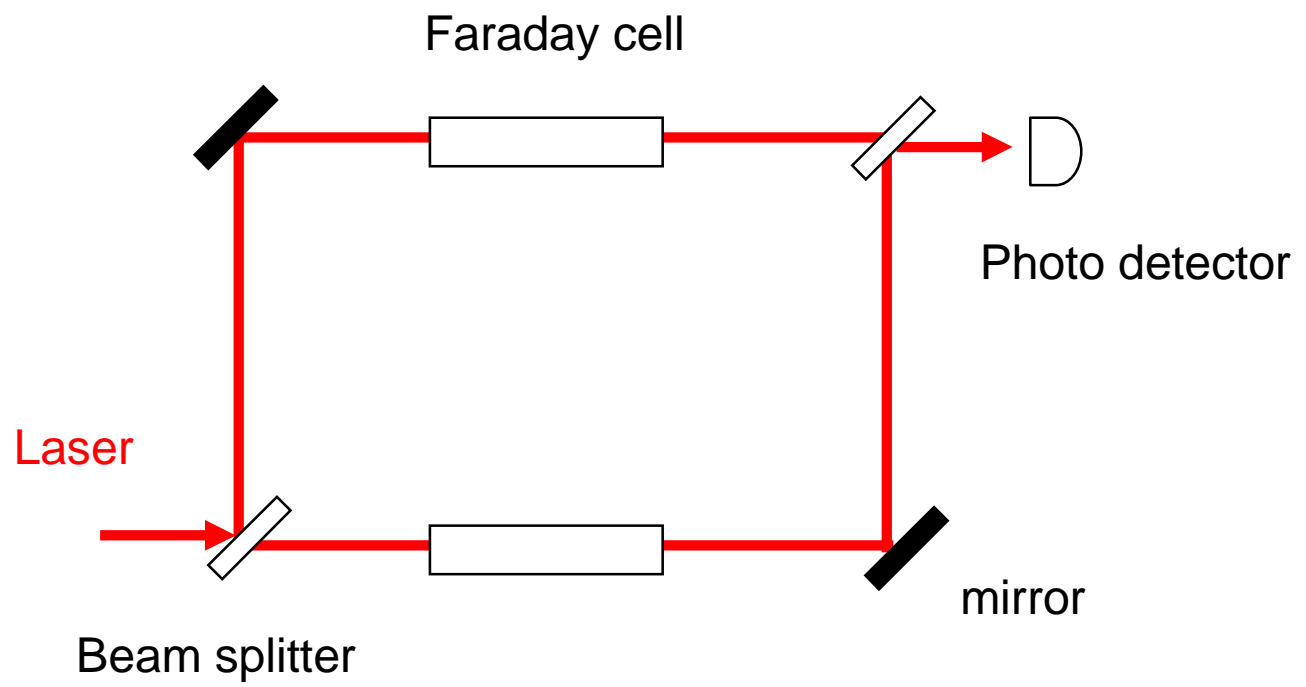


FIG. 2: Two different path ways of the light that pass through Faraday cells and are used to factorize numbers.